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C, of the tangent, at the point P, to the given helix. This tangent cuts the tangent t', which is drawn at the point P' to the circle Γ, in the horizontal trace T_1 of the required tangent. Etc.

GENOA, May–June, 1918.

DIFFERENCE QUOTIENTS.

By J. P. BALLANTINE, Cambridge, Mass.

The whole subject of interpolation, in the case of equally spaced ordinates, by means of algebraic polynomials is commonly based on the subject of finite differences. The method, however, fails for unequal spacing. The purpose of this paper is to show, that by a suitable extension of the conception of the difference quotient, the method can be generalized for the case when the ordinates are spaced according to any law.[1]

Let $f(x)$ be any single valued function. Plot the curve $y = f(x)$. Pick out on the curve the $(n+1)$ points whose x coördinates are $1, 2, \cdots, (n+1)$, respectively. By means of the method of finite differences, find the equation of the curve of the form $y = P_n(x)$ which passes through the $(n+1)$ points. For convenience, call this curve a secant of degree n, noticing that a secant of the first degree is an ordinary secant. It is easily verified that the nth derivative of this secant is precisely equal to the difference of order n of the ordinates of the $(n+1)$ points.

Now let us take our $(n+1)$ points on the curve at random. We have to define the difference quotient of order n at these $(n+1)$ points in such a way that when the values of $x_1, x_2, \cdots, x_{n+1}$ are $1, 2, \cdots, n+1$ respectively, it will reduce to a difference of order n.

We can define the secant through these $(n+1)$ points as the curve of the form $y = P_n(x)$ which passes through them. By way of a more explicit notation, $y = P_n(x;\ 1, n+1)$ will be the equation of the nth degree secant at the points $(x_1, y_1), (x_2, y_2) \cdots (x_{n+1}, y_{n+1})$. It follows that

$$P_n(x_i;\ 1,\ n+1) = y_i, \qquad 1 \leqq i \leqq n+1.$$

DEFINITION. The difference quotient of order n, taken at the $(n+1)$ points $(x_1, y_1), (x_2, y_2) \cdots (x_{n+1}, y_{n+1})$ is the nth derivative of the nth degree secant through those points. It is denoted thus $Q^n_{1,\,n+1}$:

$$Q^n_{1,\,n+1} = \frac{d^n}{dx^n} P_n(x;\ 1,\ n+1).$$

Consider the special polynomial and associated equation

$$P_{n+1}(x;\ 1,\ n+2) - P_n(x;\ 1,\ n+1),$$

$$(1) \quad P_{n+1}(x_i;\ 1,\ n+2) - P_n(x_i;\ 1,\ n+1) = y_i - y_i = 0,\ 1 \leqq i \leqq n+1.$$

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[1] Cf. A. A. Markoff, *Differenzenrechnung*, 1896, p. 10.

From the factor theorem for polynomials, we may say that:

$$(2)\quad P_{n+1}(x;\,1,n+2)-P_n(x;\,1,n+1)=K(x-x_1)(x-x_2)\cdots(x-x_{n+1}).$$

Now, determine K by $(n+1)$ differentiations:

$$(3)\qquad Q^{n+1}_{1,\,n+2}=K(n+1)!.$$

Using this value of K in (2), and transposing, we have

$$(\mathrm{I})\quad P_{n+1}(x;\,1,n+2)=P_n(x;\,1,n+1)+\frac{Q^{n+1}_{1,\,n+2}}{(n+1)!}(x-x_1)(x-x_2)\cdots(x-x_{n+1}).$$

In this way a secant of degree $(n+1)$ is expressed in terms of one of degree n and a difference quotient. We know how to write the equation of a secant of degree one or zero, and so our problem of writing the equation of a secant of the nth degree at $(n+1)$ arbitrary points is solved, except for determining the difference quotient.

In the same way as we inferred the truth of (2), we may show that

$$(2')\quad P_{n+1}(x;\,1,n+2)-P_n(x;\,2,n+2)=K(x-x_2)(x-x_3)\cdots(x-x_{n+2}).$$

It is not obvious that the two arbitrary constants must be the same, until one notices that in each case K must be equal to the coefficient of x^{n+1} in the polynomial, $P_{n+1}(x;\,1,n+2)$.

Subtracting (2′) from (2), we have

$$(4)\quad K(x_{n+2}-x_1)(x-x_2)(x-x_3)\cdots(x-x_{n+1})=P_n(x;\,2,n+2)-P_n(x;\,1,n+1).$$

Differentiating n times, we obtain

$$(5)\qquad K(x_{n+2}-x_1)(n!)=Q^n_{2,\,n+2}-Q^n_{1,\,n+1}.$$

Then by means of the value of $Q^{n+1}_{1,\,n+2}$ as obtained in (3), we have

$$(\mathrm{II})\qquad Q^{n+1}_{1,\,n+2}=\frac{Q^n_{2,\,n+2}-Q^n_{1,\,n+1}}{\dfrac{x_{n+2}-x_1}{n+1}}.$$

We are now in a position to form a table of difference quotients. Formula (II) tells us, that besides forming a difference, we must divide by a denominator, which is simply the average of the $(n+1)$ intervals of x between the $(n+2)$ points at which $Q^{n+1}_{1,\,n+2}$, the entry we are at the time computing, is taken. It may be denoted thus:

$$(6)\qquad \Delta_k x_i=\frac{x_{k+i}-x_i}{k}.$$

Applying the definition of the difference quotient of order zero, we notice that the secant of order zero through the point $(x_i y_i)$ is given by the equation: $y=y_i$. Without differentiation, we find: $Q^0_i=y_i$.

The table of difference quotients may be arranged as follows:

$$
\begin{array}{cccc|ccccccc}
 & & & x_1 & y_1 = Q_1^0 & & & & & & \\
 & & \Delta_1 x_1 & & & \Delta Q_{1,2}^0 & Q_{1,2}^1 & & & & \\
 & \Delta_2 x_1 & & x_2 & y_2 = Q_2^0 & & & \Delta Q_{1,3}^1 & Q_{1,3}^2 & & \\
\Delta_3 x_1 & & \Delta_1 x_2 & & & \Delta Q_{2,3}^0 & Q_{2,3}^1 & & & \Delta Q_{1,4}^2 & Q_{1,4}^3 \\
 & \Delta_2 x_2 & & x_3 & y_3 = Q_3^0 & & & \Delta Q_{2,4}^1 & Q_{2,4}^2 & & \\
 & & \Delta_1 x_3 & & & \Delta Q_{3,4}^0 & Q_{3,4}^1 & & & & \\
 & & & x_4 & y_4 = Q_4^0 & & & & & &
\end{array}
$$

The elements x_i and y_i are assigned.

All the elements to the left of these can be computed by relation (6) above, and those on the right by the following two relations:

$$\Delta Q_{i,\,i+k+1}^{k} = Q_{i+1,\,i+k+1}^{k} - Q_{i,\,i+k}^{k}, \tag{7}$$

$$Q_{i,\,i+k}^{k+1} = \frac{\Delta Q_{i,\,i+k+1}^{k}}{\Delta_{k+1} x_i}. \tag{8}$$

One way of writing the equation of the secant through the four points $(x_1 y_1)$, $(x_2 y_2) \cdots$ would be:

$$y = y_1 + \frac{Q_{1,2}^1}{1!}(x - x_1) + \frac{Q_{1,3}^2}{2!}(x - x_1)(x - x_2) + \frac{Q_{1,4}^3}{3!}(x - x_1)(x - x_2)(x - x_3), \tag{9}$$

obtained by a repeated application of equation (I).

In the general case the formation of a table of difference quotients involves a few laborious divisions; but these can be avoided by a proper choice of the spacing of the intervals of x. If, for instance, all of the intervals of x are unity, then $\Delta_k x_i$, the average of any k of them, will also be unity, and the rule for the formation of the difference quotients reduces to that of the formation of a table of differences. In that case the formulæ of Newton and Gauss are special cases of form (I).

These are by no means the only important special cases. Let us first consider the limiting case when one point, $(x_j,\ y_j)$, approaches the point, $(x_i,\ y_i)$. The limiting value of $Q_{i,j}^1$ would be $f'(x_i)$, if such a derivative existed, where $f(x) = y$ is the path along which the first point approached the second. The derivative may be regarded as a special case of the difference quotient, by extending the conception of the secant, as defined earlier, to take into account multiplicity. Whether regarded as a special case or as a limiting case, it may be used in formula (I) and on the right hand side of (II). Equation (II) does not hold in the special case when $Q_{1,\,n+2}^{n+1} = f^{(n+1)}(x_1)$, for if all the $(n + 2)$ points are regarded as coincident, then the denominator is zero. In that case, however, $f^{(n+1)}(x_1)$ is part of the data and does not have to be computed.

Let us illustrate the case of two coincident points by a simple example. To form a polynomial which, when x has the value 1, has the value 2 and its derivative the value 3; and when x has the value 4, has the value 5. The table of difference quotients is below.

$\Delta_2 x$	$\Delta_1 x$	x	y	Δy	Q^1	ΔQ^1	Q^2
		1	2				
	0			0	3		
$\frac{3}{2}$		1	2			-2	$-\frac{4}{3}$
	3			3	1		
		4	5				

Then

$$y = 2 + (x-1)3 + (x-1)^2\left(-\frac{4}{3}\right)\left(\frac{1}{2!}\right).$$

In the special case when all the points are coincident, the difference quotients are derivatives, and the polynomial one obtains from equation (I), is exactly the polynomial one would obtain using the same number of terms in the Taylor expansion.

Suppose we had an indefinite integral tabulated to a large number of decimal places, and wished to interpolate. For the sake of argument, suppose the table of differences would not converge rapidly enough. It often happens that we are able to compute the derivative of a function represented by an indefinite integral quite readily. It appears intuitively, that if we could in some way use to advantage the values of the derivative the process would converge more rapidly.

In the case under supposition, we know the values of a function and its derivative for equally spaced values of the argument. It is as if we knew the value of the function for equally spaced pairs of points, the two points of each pair being coincident. The difference quotient at any pair of coincident points is given to us as the derivative, and all the other difference quotients may be found by formula (II). There are so many simplifications in the process, that I will submit an example.

To find the value of $\log_{10} 31.029971$ to ten places. First we notice that it is easy to compute the derivative:

$$\frac{d}{dx}\log_{10} x = \frac{1}{x}\log_{10} e = \frac{0.4342944819}{x}.$$

We tabulate a few nearby logarithms, and the corresponding values of $\frac{0.4342944819}{x}$, as follows:

x	$\log x$	derivative
29	1. 46239 79979	0. 01497 56718
30	1. 47712 12547	0. 01447 64827
31	1. 49136 16938	0. 01400 94994
32	1. 50514 99783	0. 01357 17026
33	1. 51851 39399	0. 01316 04389.

The table may be arranged as follows:

x	$y = \log x$	Q^1	$\frac{1}{2}Q^2$	$\frac{1}{2}\Delta Q^2$
29.	1. 46239 79979	0. 01497 56718	−0. 00025 24150	0. 00000 56409
29.	1. 46239 79979	1472 32568	24 67741	1 07305
30.	1. 47712 12547	1447 64827	23 60436	51039
		1424 04391	23 09397	97248

x	$y = \log x$	Q^1	$\frac{1}{2}Q^2$	$\frac{1}{2}\Delta Q^2$
31.	1. 49136 16938	0. 01400 94994	−0. 00022 12149	0. 00001 46330
		1378 82845	21 65819	88409
32.	1. 50514 99783	1357 17026	20 77410	42183
		1336 39616	20 35227	
33.	1. 51851 39399	1316 04389		

$\frac{1}{3}Q^3$	$\frac{1}{6}Q^4$	$\frac{1}{6}\Delta Q^4$	$\frac{1}{5}Q^5$
0. 00001 12818	−0. 00000 05513	0. 00000 00286	0. 00000 00858
1 07305	5227	396	792
1 02078	4830	243	729
97248	4588	336	672
92660	4251	207	621
88409	4043		
84366			

The entries in the column headed y are the data as copied from some table. Notice that the second entry is the same as the first, for they are the values of the function for two coincident values of x. This repetition is superfluous, and may be omitted. The column headed Q^1 contains the difference quotients at every two consecutive points. The first two values of x are coincident, so the first difference quotient is copied from the table of derivatives. The second and third values of x are a unit apart, so the difference quotient is the difference. And so on down the column, alternate entries are copied from the table of derivatives, and the others are obtained as differences of y.

In order to obtain Q^2, formula (II) tells us to divide the difference of two consecutive entries in the column of first order difference quotients by $\frac{1}{2}$. If you have a set of intervals, alternate ones zero and one, the average of any two consecutive ones of them will be $\frac{1}{2}$. Let us omit dividing by this factor, and thereby obtain $\frac{1}{2}Q^2$ instead of Q^2. To obtain the entries in the column headed $\frac{1}{2}\Delta Q^2$, we difference those of the column before. Formula (II) says that in order to obtain Q^3 we must divide the first ΔQ^2 by 1/3, and the second by 2/3. The differences of x are 0, 1, 0, 1, etc., and the average of three consecutive intervals will be 1/3 and 2/3 alternately. In order to avoid labor, we multiply every other one by 2, for they look smaller than the others, and thereby in each case obtain $\frac{1}{3}Q^3$ instead of $\frac{1}{2}Q^3$. Absorb these factors into the difference quotients, denoting the pseudo difference quotients by $Q_1{}^n$. The factorial divisors in the terms of the polynomial will be changed, as follows:

$$
\begin{aligned}
F(t) = y_1 &+ \frac{tQ_1}{(0!)^2(1!)} + \frac{t^2Q_1{}^2}{(0!)^2(1!)^2} + \frac{t^2(t-1)Q_1{}^3}{(1!)^2(2!)} + \frac{t^2(t-1)^2Q_1{}^4}{(1!)(2!)^2} + \cdots \\
&+ \frac{t^2(t-1)^2(t-2)^2\cdots(t-n+1)^2(t-n)}{(n!)^2(n+1)!}\,Q_1{}^{2n+1} \qquad (10) \\
&+ \frac{t^2(t-1)^2(t-2)^2\cdots(t-n)^2}{(n!)((n+1)!)^2}\,Q_1{}^{2n+2}, \qquad (t = x - x_1).
\end{aligned}
$$

We obtain in this particular example the polynomial:

$$y = 1.4913616938 + 0.0140094994t - 0.000221212149t^2$$
$$+ \frac{t^2(t-1)}{2}0.0000092660 - \frac{t^2(t-1)^2}{4}0.0000004251 + \cdots.$$

To obtain log 31.029971, set $t = 0.029971$; adding the various terms of the polynomial, we obtain the result

$$\begin{array}{r}
1.\ 49136\ 16938 \\
41\ 98787 \\
-1995 \\
-41 \\
-1 \\
\hline
1.\ 49178\ 13688
\end{array} = \log 31.029971.$$

Although the definition given above for the secant is geometrical, it could just as well be stated analytically. The work which I have carried out for reals could be extended into the domain of imaginaries.

When one has obtained the interpolatory polynomial, the secant, it may be asked, "How near will this come to giving the true value of the function throughout a certain interval?"

In equation (I), set $x = x_{n+2}$, and replace $P_{n+2}(x_{n+2};\ 1, n+2)$ by its equivalent value $f(x_{n+2})$. Considering x_{n+2} as the independent variable, and calling it x for simplicity, we get the equation:

$$\text{(III)}\quad f(x) \equiv P_n(x;\ 1, n+1) + \{Q^{n+1}_{1,n+1}(x)\}(x - x_1)(x - x_2)\cdots(x - x_{n+1})\frac{1}{(n+1)!}$$

where $Q^{n+1}_{1,n+1}(x)$ is the difference quotient taken for the $(n+1)$ fixed values $x_1, x_2, \cdots, x_{n+1}$, and the one variable value, x. It is a function of x.

A *remainder* is that which has to be added to an approximation in order to give the true result.

Let

$$f(x) = P_n(x;\ 1, n+1) + R(x).$$

Then

$$\text{(IV)}\qquad R(x) = \frac{1}{(n+1)!}\{Q^{n+1}_{1,n+1}(x)\}(x - x_1)(x - x_2)\cdots(x - x_{n+1}).$$

We have determined $R(x)$ in terms of $Q^{n+1}_{1,n+1}(x)$, and so let us study the latter. Let us examine its derivative. By the use of formula (II), we have

$$\left.\frac{dQ^{n+1}_{1,n+1}(x)}{dx}\right|_{x=x_0} = \operatorname*{Lim}_{x_{n+2}\to x_0} \frac{Q^{n+1}_{1,n+2} - Q^{n+1}_{0,n+1}}{x_{n+2} - x_0} = \operatorname*{Lim}_{x_{n+2}\to x_0} \frac{Q^{n+2}_{0,n+2}}{n+2}.$$

In general there is no limit, but let us suppose $f'(x)$ exists at $x = x_0$. Then the secant at the points $x_0, x_1, \cdots, x_{n+2}$ approaches a limiting secant as x_{n+2} approaches x_0, namely the secant at the points $x_0, x_1, \cdots, x_{n+1}$, which has contact at x_0. Its derivative of order $(n+2)$, the difference quotient, has a limiting value also, and is denoted thus: $Q^{n+2}_{0,0,n+2}$. Then we have

$$\text{(V)} \qquad \left.\frac{dQ^{n+1}_{1,n+1}(x)}{dx}\right|_{x=x_0} = \frac{Q^{n+2}_{0,0,n+2}}{n+2}.$$

Proceeding in this way one can determine the properties of $Q^{n+1}_{1,n+1}(x)$ from those of $f(x)$, and in that way can obtain the oscillation of the remainder for any interval. If the problem is to determine the derivative of $f(x)$, one differentiates in formula (III), and obtains the expression for the remainder in that case in terms of $Q^{n+1}_{1,n+1}(x)$ and its derivative, which are already known.

In this way, the difference quotient enables us to expand an arbitrary function, analytic or merely continuous, real or complex, at points to suit the peculiar needs of the problem. Moreover, in any special case, in which a simpler and less elastic method of expansion can be used, the expansion in terms of difference quotients reduces automatically to that method. The remainder can be expressed explicitly in terms of the difference quotient, not as in the case of a remainder in terms of a derivative at some unknown point, but taken at the points about which we are expanding and the one additional point x, the point at which we want the remainder.

A THEOREM IN THE GEOMETRY OF THE TRIANGLE.

By J. W. CLAWSON, Ursinus College.

It is the purpose of this paper to state a general theorem of which the well-known Wallace (or Simson[1]) line theorem is a very special case; to point out the result of a polar reciprocation; to mention some special cases of the theorem; and to give an application to the geometry of the quadrilateral.

1. *Theorem.*[2] $A_1A_2A_3$ is a triangle, a_i the side opposite A_i, $(i = 1, 2, 3)$, P a point on the circumference of the circumscribed circle[3] C, Q any point in the plane. Let A_iQ cut the circle[3] at U_i; PU_i intersects a_i at L_i. Then L_1, L_2, L_3, Q are collinear on l. Let us call this line the "P-transversal of Q" with respect to the triangle $A_1A_2A_3$.

The theorem follows at once from Pascal's Theorem for, since $A_2A_3A_1U_1PU_2$, $A_1A_2A_3U_3PU_1$ are inscribed hexagons, L_1, L_2, Q and L_3, L_1, Q are collinear.

Cor. 1.—If a transversal cut the side a_i of a triangle at L_i, and P be any point on the circumcircle, and if PL_i cut the circle at U_i, then A_1U_1, A_2U_2, A_3U_3 are concurrent at Q, a point on the transversal.

Cor. 2.—If a transversal cut the side a_i of a triangle at L_i, if Q is any point

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[1] For references, see this MONTHLY, 1916, p. 61.

[2] It would be strange if this simple theorem were new; yet the writer has not found it recorded. A special case is given in Hatton's *Projective Geometry*, 1913, Ex. p. 164.

[3] Or conic.